$$\begin{array}{ccc} X=\{x_1,\dots,x_n\} & \rightarrow & G \\ f\searrow & & \swarrow \bar{f} \\ & H & \end{array}$$

Then we define $\bar{f}\colon G \to H$ by $\bar{f}(x_i) = h_i$ and extend it by using $\mathbb{Z}$-linearity.

(2.2.2) Let G be an (additive) abelian group with the identity element 0. An element $g \in G$ is said to be *torsion* if there is an integer n such that $ng = 0$. We say that G is a *torsion group* if every element of G is torsion; G is *torsion free* if every non-zero element of G is not a torsion. Define

$$G_\tau = \{g \in G \mid g \text{ is torsion}\}.$$

Then G_τ is a torsion group, and G/G_τ is torsion free. We have an exact sequence

$$(0) \longrightarrow G_\tau \longrightarrow G \longrightarrow G/G_\tau \longrightarrow (0)$$

where G_τ is a torsion group and G/G_τ is torsion free.

(2.2.3) *Let G, G_1 and G_2 be abelian groups. Then the following conditions are equivalent.*

(i) $G \cong G_1 \oplus G_2$.

(ii) *We have an exact sequence*

$$(*) \qquad (0) \longrightarrow G_1 \xrightarrow{\iota} G \xrightarrow{\pi} G_2 \longrightarrow (0)$$

and there is a group homomorphism $s\colon G_2 \to G$ (called a *section*) *such that* $\pi \circ s = \mathrm{id}$, i.e., *the above exact sequence splits.*

(iii) *We have an exact sequence* (*) *of* (ii) *and a group homomorphism* $r\colon G \to G_1$ (called a *retraction*) *such that* $r \circ \iota = \mathrm{id}$.

(iv) *There are endomorphisms $\phi_i\colon G \to G$ $(i = 1, 2)$ such that*

$$\mathrm{Im}(\phi_i) = G_i, \phi_1 + \phi_2 = \mathrm{id}_G \quad \text{and} \quad \phi_i \circ \phi_j = \delta_{ij}\phi_j$$

where δ_{ij} is the Kronecker delta.

Furthermore, if G_2 is a free abelian group in (*) *of* (ii) *then the exact sequence splits.* (See Ex. 1 for further equivalent conditions.)

Proof. (i) $\Rightarrow$ (ii) By identifying G with $G_1 \oplus G_2$ we choose ι to be the inclusion of G_1 into G and π be the projection to the second factor. Now define $s(x) = (0, x)$.

(ii) $\Rightarrow$ (iii) Let $x \in G$. Then $x - s \circ \pi(x)$ is in the kernel of π which is the same as the image of ι. Since ι is injective, there is a unique $y \in G_1$ such that $\iota(y) = x - s \circ \pi(x)$. Now define $r(x) = y$. Now one checks that $r \circ \iota = \mathrm{id}$.

(iii) $\Rightarrow$ (i) Define a map $G \to G_1 \oplus G_2$ by sending an element x of G to $(r(x), \pi(x))$. For the inverse of this map let $(a, b) \in G_1 \oplus G_2$ and let $b' \in G$ be such that $\pi(b') = b$. Now map (a, b) to $\iota(a) + b' - \iota \circ r(b')$. One checks that these maps are inverses to each other.

(i) $\Rightarrow$ (iv) Let ϕ_i be the composition $G \xrightarrow{proj.} G_i \xrightarrow{incl.} G$. Now it is easy to check (iv).

(iv)$\Rightarrow$ (i) Exercise.

For the last part, let $\{z_i\}$ be a free basis of G_2 and x_i be a lift of z_i in G. Then there must be a torsion free element in the coset $x_i + G_1$ in G, say $\bar{x}_i$ (otherwise $\pi(x_i) = z_i$ must be a torsion). Hence we can define the map s by requiring $s(z_i) = \bar{x}_i$.

We remark here that if the groups are non-abelian then the results above are false. For example, consider a semidirect product $N \times_\phi H$. We have an exact sequence

$$(e) \longrightarrow N \xrightarrow{\iota} N \times_\phi H \xrightarrow{\pi} H \longrightarrow (e).$$

There is a section $s\colon H \to N \times_\phi H$ defined by $s(h) = (0, h)$. However, $N \times_\phi H$ is not isomorphic to the direct sum $N \oplus H$ unless ϕ is trivial. Cf. Ex.2.1.6.

(2.2.4) *A subgroup H of a free abelian group G of rank n is free of rank $\le n$. A finitely generated torsion free abelian group is free.*

Proof. We induct on n. The result for $n = 1$ is well known. (A subgroup of $\mathbb{Z}$ is of the form $n\mathbb{Z}$ for some integer n and it is isomorphic to $\mathbb{Z}$.) Let $G = \bigoplus_{i=1}^{n} \mathbb{Z}x_i$ $(n > 1)$. Consider